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CONCEPTS AND EQUATIONS FOR MULTILAYERED  
VARIABLE DENSITY ESTUARINE HYDRAULICS

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May 1971

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Concepts and Equations  
For Multilayered, Variable  
Density Estuarine Hydraulics

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Technical Report No. 8

### Introduction:

The versatility of the "channel/junction" hydrodynamic and quality models is well established. Their abilities as predictive tools have been verified in many widely varying locations.

However, there are two basic assumptions inherent in the models that limit their uses. The first and primary assumption both hydraulically and for quality is that of complete mixing vertically in the junctions. The second assumption is that the junctions are connected with channels that are horizontal.

It is the aim of this paper to present the derivation of the equations of motion and continuity for a multilayered system of channels and junctions in which channels slopes and density differences play an integral part.

Only with such a system may the effects of the vertical placement of heated discharges or of subsurface currents on contaminant distribution be predicted.

1.

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Horizontal forces:

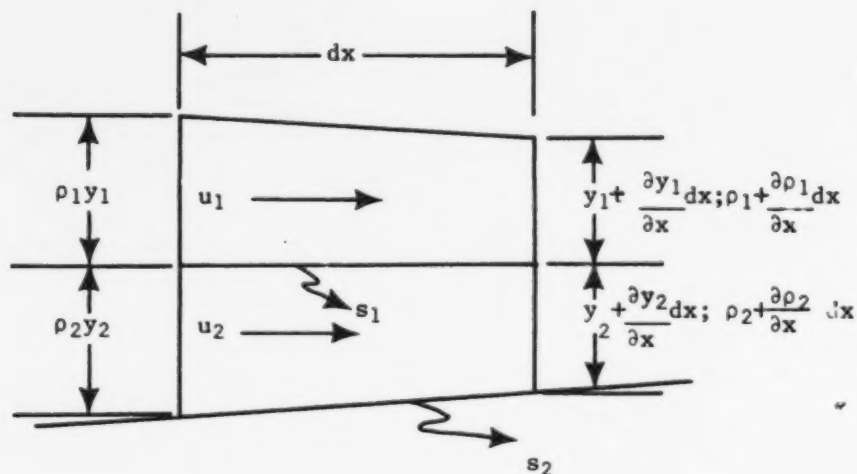


figure A.

Consider the two layered slice  $dx$  long of width  $B$ . (figure A)

Assuming  $\frac{\partial y_1}{\partial x} dx$  to be very small compared to  $y_1$ , the mass of each layer is

$$M_1 = \rho_1 y_1 B dx \quad (1)$$

$$M_2 = \rho_2 y_2 B dx \quad (2)$$

Each layer is acted upon by the forces of gravity, friction, pressure and inertia. These forces must sum to zero. The forces acting on layer (1) are:

Inertia:

$$F_I = \rho_1 y_1 B dx \left( \frac{du_1}{dt} \right) \quad (3)$$

or

$$F_I = \rho_1 y_1 B dx \left( \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} \right) \quad (4)$$

Pressure:

$$F_P = \frac{\rho_1 g B}{2} y_1^2 - \left( \frac{\rho_1}{2} \frac{\partial \rho_1}{\partial x} dx \right) g B \left( y_1 + \frac{\partial y_1}{\partial x} dx \right)^2 \quad (5)$$

which reduces to

$$F_P = \frac{-g B dx}{2} \frac{\partial (\rho_1 y_1^2)}{\partial x} \quad (6)$$

Gravity:

$$F_G = +\rho_1 y_1 B S_1 dx g \quad (7)$$

Friction:

$$F_F = B dx (\tau_o + \tau_1) \quad (8)$$

where  $\tau$  is the shear stress on the layer bottom and  $\tau_o$  is the stress on the layer top.

In general  $\tau_i$  is assumed to be quadratic. ( $\tau_i = \rho_i f u_i^2$ ) where  $f$  is assumed to be independent of the variation in  $u$  with respect to time and  $u$  is the relative motion between layers.

$$\rho_1 y_1 B dx \left( \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} \right) = -\rho_1 y_1 B S_1 dx g - \frac{g B dx}{2} \frac{\partial (\rho_1 y_1^2)}{\partial x} - B dx (\tau_o + \tau_1) \quad (11)$$

Divide by  $\rho_1 y_1 B dx$  to get

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = -s_1 g - \frac{g}{\partial \rho_1 y_1} \frac{\partial (\rho_1 y_1^2)}{\partial x} - \frac{(\tau_0 + \tau_1)}{\rho_1 y_1} \quad (12)$$

Note that if  $\tau_1$  is expressed in terms of the Chezy coefficient

and  $\tau_0 = 0$

$$\tau_1 = \rho_1 \frac{g}{c^2} u/u/ \quad (13)$$

and  $\rho_1$  is assumed constant, equation (12) reduces to the equation of Barre' St. Venant.

$$\frac{\partial u_1}{\partial t} + u \frac{\partial u_1}{\partial x} + g \frac{\partial h}{\partial x} = -g S_1 - \frac{g}{C_h^2} \frac{u/u/}{R_h} \quad (14)$$

where  $R_h$  is the hydraulic radius of the channel, approximately equal to  $y_1$  for a wide channel.

The forces exerted on block (2) are arrived at in the same manner.

The pressure force is handled by defining pressures.

$$P_a = \rho_1 g y_1 \quad (15)$$

and

$$P_a + \frac{\partial P_a}{\partial x} dx = \left( \rho_1 + \frac{\partial \rho_1}{\partial x} \right) g \left( y_1 + \frac{\partial y_1}{\partial x} dx \right) \quad (16)$$

Thus the resultant pressure force on (2) is

$$F_p = \left[ P_{a1} + \frac{\rho_2 g y_2}{2} \right] y_2 B - \left[ \left( P_{a1} + \frac{\partial P_{a1}}{\partial x} dx \right) + g \left( \frac{\rho_2 \frac{\partial \rho_2}{\partial x} dx}{2} y + \frac{\partial y_2}{\partial x} dx \right) \right] \left( y_2 \frac{\partial y_2}{\partial x} \right) B \quad (17)$$

which reduces to

$$F_p = \frac{-Bgdx}{2} \left[ \frac{2}{g} \frac{\partial (y_2 P_{a1})}{\partial x} + \frac{\partial (y_2^2 \rho_2)}{\partial x} \right] \quad (18)$$

Summing all the horizontal forces on (2) gives

$$\left[ \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} \right] = -S_2 g \frac{g}{\partial \rho_2 y_2} \left[ \frac{2}{g} \frac{\partial (y_2 P_{a1})}{\partial x} + \frac{\partial (y_2^2 \rho_2)}{\partial x} \right] - \frac{(\tau_1 + \tau_2)}{\rho_2 y_2} \quad (19)$$

Generalizing equation (19) to a system of N layers, the equation of motion can be expressed as

$$\left[ \frac{\partial u_n}{\partial t} + u_n \frac{\partial u_n}{\partial x} \right] = -S_n g - \frac{g}{\partial \rho_n y_n} \left[ \frac{2}{g} \frac{\partial (y_n P_{a_{n-1}})}{\partial x} + \frac{\partial (y_n^2 \rho_n)}{\partial x} \right] - \frac{(\tau_{n-1} + \tau_n)}{\rho_n y_n} \quad (20)$$

$n = 1, 2, 3, \dots, N$

Note that at the first layer (i.e., at the air/water interface)  $\tau_0$  represents wind stress and  $\frac{\partial P_a}{\partial x}$  represents the variation of atmospheric pressure along the channel.

Further discussion of the term  $S_1$  and its relationship to the depth  $y_1$  is needed.

Define  $h_i$  as the elevation of the layer interface at any time  $t$  and at any point  $x$ .

The slope of the interface and the quantities  $y_i$  and  $h_i$  are related by the expression.

$$\frac{\partial y_i}{\partial x} = -S_i + \frac{\partial h_i}{\partial x} \quad (21)$$

Successive layers are related by

$$\frac{\partial h_i}{\partial x} = S_{i-1} \quad (22)$$

The shear stress terms  $\tau_i$  are very important to the distribution of flow and will be discussed at length in a later section.

Vertical forces:

The vertical forces acting on the layers of the channel slice fall into two categories according to their origin.

The first force is the local vertical acceleration causing the stream lines along  $dx$  to bend. At the free surface this acceleration is maximum with a value of

$$A_v = \frac{\partial^2 h}{\partial t^2} \quad (23)$$

and the vertical velocity  $v_{MAX}$  is expressed by

$$v_{MAX} = \frac{\partial h}{\partial t} \quad (24)$$

The Eulerian equations of motion give the vertical acceleration as

$$A_v = -\frac{1}{\rho} \frac{\partial P}{\partial Z} - g \quad (25)$$

where  $P$  is pressure

and  $Z$  is the vertical coordinate. If the pressure distribution is hydrostatic, the right hand side of equation (25) vanishes and no vertical velocities may exist.

At any rate, in a periodic wave, the net transport vertically will be zero. Thus vertical pressure variations from hydrostatic will be ignored.

Equations (23) and (24) do provide a logical and convenient basis of computing the vertical dispersion characteristics of the channel.

The vertical velocity at any depth may be derived from (24) by assuming a linear distribution as

$$v_{(Z)} = \frac{Z}{y} \frac{\partial h}{\partial t} \quad (26)$$

and the vertical dispersion  $E_v$  may be formulated as

$$E_v = f(v_{MAX}(Z)) \quad (27)$$

and may be determined empirically from observed data.

In the second category of vertical forces is the buoyant force; either positive or negative. Returning to block (1) in figure A,

the buoyant force is the resultant of the forces of

Gravity:

$$F_G = \rho_l g y_l B dx \quad (28)$$

and

Pressure:

$$F_p = \left[ P_{a1} + \left( P_{a1} + \frac{\partial P_{a1}}{\partial x} dx \right) \right] \frac{B dx}{2} - \left[ P_{a0} + P_{a0} + \frac{\partial P_{a0}}{\partial x} dx \right] \frac{B dx}{2} \quad (29)$$

which reduces to

$$F_p = \frac{B dx}{2} \left[ 2P_{a1} + \frac{\partial P_{a1}}{\partial x} dx - 2P_{a0} - \frac{\partial P_{a0}}{\partial x} dx \right] \quad (30)$$

The inertia force is expressed as

$$F_I = \rho_l y_l B dx \frac{dw_l}{dt} \quad (31)$$

where  $W_l$  is the vertical velocity positive in the upward direction.

Vertical shear has been neglected due to the small velocities involved.

Summing forces vertically:

$$F_I + F_p + F_G = 0 \quad (32)$$

$$\rho_l y_l B dx \frac{dw_l}{dt} = -\rho_l y_l B dx g + \frac{B dx}{2} \left[ 2P_{a1} + \frac{\partial P_{a1}}{\partial x} dx - \left( 2P_{a0} + \frac{\partial P_{a0}}{\partial x} dx \right) \right] \quad (33)$$

and dividing by  $\rho_l y_l B dx$  gives:

$$\frac{dw_l}{dt} = -g + \frac{1}{2\rho_l y_l} \left[ 2P_{a1} + \frac{\partial P_{a1}}{\partial x} dx - 2P_{a0} - \frac{\partial P_{a0}}{\partial x} dx \right] \quad (34)$$

Generalizing for the  $n$ th layer of an  $N$  layer system,

$$\frac{dw_n}{dt} = -g + \frac{1}{2\rho_n y_n} \left[ 2P_{a_n} + \frac{\partial P_{a_n}}{\partial x} dx - 2P_{a_{n-1}} - \frac{\partial P_{a_{n-1}}}{\partial x} dx \right] \quad (35)$$

and the  $P_a$  terms are as defined in equations (15) and (16)

Friction forces and shear stresses:

Von Karman's modification to Prandtl's mixing length theory of shear stress in turbulent flow gives shear stress at any point  $Z$  in the vertical as

$$\tau = \rho \kappa^2 \left( \frac{du}{dz} \right)^2 \quad (36)$$

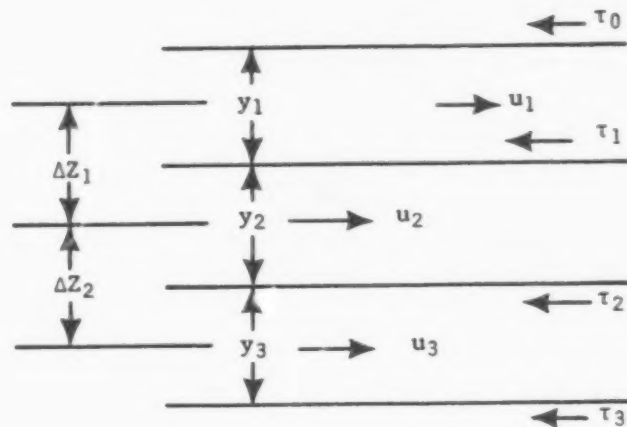


figure B.

where  $\kappa$  is a universal constant equal to 0.4

Applied to the three layered system of figure B,

$$\tau_n = \rho_n \kappa^2 \frac{\left(\frac{du_n}{dz}\right)^4}{\left(\frac{d^2u_n}{dz^2}\right)^2} \quad (37)$$

In finite difference form

$$\frac{du_n}{dz} = \frac{u_n - u_{n-1}}{\Delta Z_{n-1}} \quad (38)$$

and

$$\frac{d^2u_n}{dz^2} = \frac{\frac{u_n - u_{n-1}}{\Delta Z_{n-1}} - \frac{u_n - u_{n+1}}{\Delta Z_n}}{\Delta Z_{n-1} + \Delta Z_n} \quad (39)$$

$$\text{where } \Delta Z_{n-1} = \frac{y_{n-1}}{2} + \frac{y_n}{2} \quad (40)$$

$$\text{and } \Delta Z_n = \frac{y_n}{2} + \frac{y_{n+1}}{2} \quad (41)$$

Then

$$\tau_n = \rho_n \kappa^2 \left[ \frac{u_n - u_{n-1}}{\Delta Z_{n-1}} \right]^4 \left/ \left[ \frac{\frac{u_n - u_{n-1}}{\Delta Z_{n-1}} - \frac{u_n - u_{n+1}}{\Delta Z_n}}{\Delta Z_{n-1} + \Delta Z_n} \right]^2 \right. \quad (42)$$

where  $\tau_n$  acts to make velocity differences between layers smaller.

Equations (20) and (42) can be combined to describe the horizontal motion of the fluid in the elemental slice. Equation (42) gives an approximate picture of the interchange between layers due to buoyant effects.

The velocity gradient  $\frac{\partial u_n}{\partial x}$  in equation (20) can be transformed by the use of the continuity equation;

$$\frac{\partial A}{\partial t} + \frac{\partial(AU)}{\partial x} = 0 \quad (43)$$

where A is the cross section yB and B is the constant width.

$$\frac{\partial A}{\partial t} + u \frac{\partial A}{\partial t} + A \frac{\partial u}{\partial x} \quad (44)$$

$$\frac{\partial U}{\partial x} = -\frac{1}{A} \left[ \frac{\partial A}{\partial t} + u \frac{\partial A}{\partial x} \right] \quad (45)$$

and since  $A=yB$  ;  $\frac{\partial A}{\partial x} = B \frac{\partial y}{\partial x}$

and

$$\frac{\partial u_n}{\partial x} = -\frac{1}{A_n} \frac{\partial A_n}{\partial t} - \frac{u_n}{B} \frac{\partial y_n}{\partial x} \quad (46)$$

where again

$$\frac{\partial y_n}{\partial x} = -S_n + \frac{\partial h_n}{\partial x} \quad (47)$$

Substitution of equation (47) into equation (20) makes a more tractable equation for numerical solution.

$$\begin{aligned} \frac{\partial u_n}{\partial t} = & \frac{1}{A_n} \frac{\partial A_n}{\partial t} + \frac{u_n}{B} \frac{\partial y_n}{\partial x} - Sng \frac{-g}{\partial \rho_n y_n} \left[ \frac{2}{g} \frac{\partial (y_n^p a_{n-1})}{\partial x} + \right. \\ & \left. \frac{\partial (y_n^2 \rho_n)}{\partial x} \right] - \frac{(\tau_{n-1} + \tau_n)}{\rho_n y_n} \end{aligned} \quad (48)$$

The finite element forms of the equations:

Equation (48) can be expressed as

$$\frac{\Delta U_n}{\Delta t} = \frac{1}{A_n} \frac{\Delta A_n}{\Delta t} + \frac{u_n}{B} \frac{y_n}{L} - s_n g - \frac{g}{\partial \bar{\rho}_n d_n} \left[ \frac{2}{g} \frac{\Delta(y_n^2 P_{an-1})}{L} + \frac{\Delta(y_n^2 \rho_n)}{L} \right] - \frac{(\tau_{n-1} + \tau_n)}{\bar{\rho}_n d_n} \quad (49)$$

where

$d_n$  = layer thickness at the channel midpoint

$\bar{\rho}_n$  = average density in the channel

Equation (35) becomes

$$\frac{\Delta W_n}{\Delta t} = -g + \frac{1}{2\rho_n y_n} \left[ \partial P_{an} + \Delta P_{an} - \partial P_{an-1} - \Delta P_{an-1} \right] \quad (50)$$

The continuity equation is written as

$$\frac{\Delta y_n}{t} = \frac{\Sigma \text{Inflows} - \Sigma \text{Outflows}}{A_{sn}} \quad (51)$$

Figure C represents three junctions, I, J, K connected by channels M, N. There are three layers in this example.

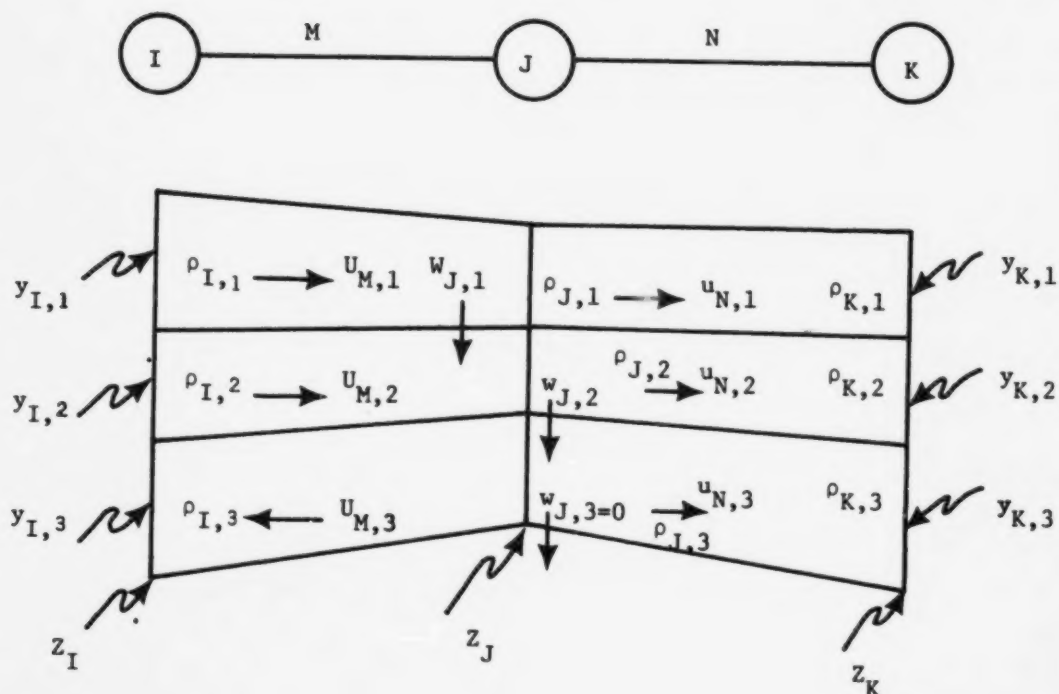


figure C.

Symbols retain the same meaning, but will now be doubly subscripted. (i.e.,  $y_{I,l}$ , refers to the thickness of layer  $l$  at junction I, and  $u_{N,l}$  refers to the velocity in layer  $l$  in channel N.)

Variables are separated into junction variables and channel variables as follows.

Channel variables:

|                    |  |                                                          |
|--------------------|--|----------------------------------------------------------|
| $U_{N,l}$          |  | horizontal velocity                                      |
| $Q_{N,l}$          |  | horizontal flow                                          |
| $\tau_{N,l}$       |  | shear stress                                             |
| $A_{N,l}$          |  | layer crosssectional area at the<br>channel midpoint     |
| $d_{N,l}$          |  | layer thickness at the channel<br>midpoint               |
| $B_N$              |  | channel width (constant for all<br>layers)               |
| $L_N$              |  | channel length (constant for all<br>layers)              |
| $S_{N,l}$          |  | slope of the interface between layers<br>in the channel. |
| $\bar{\rho}_{N,l}$ |  | average channel density                                  |

Junction variables:

|              |   |                                           |
|--------------|---|-------------------------------------------|
| $w_{I,l}$    | = | vertical velocity                         |
| $q_{I,l}$    | = | vertical flow                             |
| $H_{I,l}$    | = | layer head                                |
| $\rho_{I,l}$ | = | density                                   |
| $z_I$        | = | bottom elevation above some assumed datum |

|            |   |                                           |
|------------|---|-------------------------------------------|
| $A_{SI}$   | = | surface area                              |
| $y_{I,l}$  | = | layer thickness                           |
| $V_{I,l}$  | = | layer volume                              |
| $P_{aI,l}$ | = | applied pressure at the top of the layer. |

( $P_{aI,0}$  represents the atmospheric pressure).

Certain basic relationships between the variables can be expressed as:

$$Q_{N,l} = (U_{N,l}) (A_{N,l}) \quad (52)$$

$$A_{N,l} = (d_{N,l}) (B_N) \quad (53)$$

$$S_{N,l} = (H_{J,l} - H_{K,l}) / L_N \quad (54)$$

$$q_{I,l} = (W_{I,l}) (A_{SI}) \quad (55)$$

$$H_{I,l} = Z_I + H_{I,l+1} + y_{I,l} \quad (56)$$

$$P_{aI,l} = P_{aI,l-1} + \rho_{I,l} y_{I,l} g \quad (57)$$

$$d_{N,l} = (y_{J,l} + y_{K,l}) / \partial \quad (58)$$

$$\bar{\rho}_{N,l} = \frac{(y_{J,l} \rho_{I,l} + y_{K,l} \rho_{K,l})}{(y_{J,l} + y_{K,l})} \quad (59)$$

$$V_{I,l} = (y_{I,l}) (A_{SI}) \quad (60)$$

### Density computations:

Normally when a fluid is said to be incompressible, the assumption is made that the fluid density is constant. However incompressibility means that volumes do not change with pressure, not that density is constant.

### Conservation of density:

Since, in the multilayered equations of motion, density is an integral part, the time and spatial variations must be computed simultaneously with flow.

This is different from the normal method of computing the hydraulics of a system and the quality separately.

The driving force for the density currents in most estuaries is salinity. That is to say horizontal salinity gradients causes a sloping of the isopleths of pressure with distance.

In this treatment density will be considered to be a conservative quantity with the same characteristics as TDS or chloride. Thus the rate of change in density of a layer is merely the sum of all the rates of contribution or extraction by advection and dispersion.

$$\frac{\Delta \rho_{J,L}}{\Delta t} = \left( \sum Q \rho_{in} - \sum Q \rho_{out} + \sum E_H A_s \frac{\Delta \rho}{L} - \sum E_V A_s \frac{\Delta \rho}{y} \right) / V_{J,L} \quad (61)$$

where  $E_H$  and  $E_V$  are horizontal and vertical dispersion coefficients.

In figure C, equation (61) can be written for layer 2 of junction J as:

$$\frac{\Delta \rho_{J,2}}{\Delta t} = \left[ (Q_{M,2} \rho_{I,2}) - (Q_{N,2} \rho_{J,2}) + (q_{J,1} \rho_{J,1}) - (q_{J,2} \rho_{J,2}) \right. \\ \left. - \left( \frac{\rho_{I,2} - \rho_{J,2}}{L_M} A_{M,2} E_{HM,2} \right) - \left( \frac{\rho_{J,2} - \rho_{K,2}}{L_N} A_{N,2} E_{HN,2} \right) \right. \\ \left. - \left( \frac{\rho_{J,2} - \rho_{J,1}}{y_{J,1} + y_{J,2}} \right) A_{SJ} E_{VJ,2} - \left( \frac{\rho_{J,2} - \rho_{J,3}}{y_{J,2} + y_{J,3}} \right) A_{SJ} E_{VJ,2} \right] / v_{J,2} \quad (62)$$

The dispersion coefficients can be determined empirically from observed data. Since the time step length is generally quite small, dispersion is probably not a significant factor in the density calculation.

#### Solution of the equations:

Solution of the horizontal motion equation (49), the vertical motion equation (50), the continuity equation (51) and the density equation (61) must be done by a numerical integration technique,

subject to the following boundary and initial conditions.

The variables of at least one junction must be known for all time. This junction is called the driving head and is normally the downstream or ocean end of the estuary. At all other junctions only the flow rates, and the density of the inflow need be known.

Initial values of the variables at each junction are required. The accuracy of their determination will affect the rate of convergence of the solution, but will not affect the final values.

The stability criterion for a one dimensional solution of the hydrodynamic equations is that

$$\sqrt{gd} < \frac{L}{\Delta t} \quad (63)$$

where D is the depth in the channel.

This equation relates the celerity of a long wave in shallow water to the channel length and travel time.

To take into account the tidal current as well as wave celerity equation (63) can be written as

$$| V + \sqrt{gd} | < \frac{L}{\Delta t} \quad (64)$$

In the stratified case V in equation (64) contains both horizontal and vertical velocity. or:

$$| \sqrt{u^2 + w^2} + \sqrt{gd} |_{\text{MAX}} < \frac{L}{\Delta t} \quad (65)$$

A complicating factor in the choice of time step and channel length is the problem of numerical mixing or numerical dispersion. This is a common problem in the solution of differential equations by finite elements and criteria have been established which can minimize the effect.

Stated for this problem the criteria is

$$\left( \frac{\sqrt{u^2 + w^2}}{\sqrt{L^2 + Z^2}} \right) \Delta t = 1 \quad (66)$$

Assume W and Z are sufficiently small compared to U and L so that the equation reduces to

$$\phi = \frac{u \Delta t}{L} = 1 \quad (67)$$

Examination of equations (67) and (64) shows that the two differ by the wave celerity term

$$\phi + \frac{\sqrt{gd} \Delta t}{L} = 1 \quad (68)$$

By defining two separate time steps ( $\Delta t_h$ ) and ( $\Delta t_p$ ) for the hydraulic and densimetric solutions respectively, equation (68) can be rearranged to give the ratio

$$\frac{\Delta t_p}{\Delta t_h} = \left| \frac{1}{U} \left( \frac{L}{\Delta t_h} - \sqrt{gd} \right) \right| \quad (69)$$

If the layer depth ( $d$ ) is assumed to be fairly constant over a tidal cycle the terms within the parentheses will be constant.

$$\left( \frac{L}{\Delta t_n} - \sqrt{gd} \right) = K \quad (70)$$

and the ratio ( $T_R$ ) will be a function of advective velocity and the constant  $K$ .

$$T_R = K/U \quad (71)$$

The term  $T_R$  can range from a very large number ( $U \rightarrow 0$ ) to a value of 1 during the tidal cycle. In addition, in large scale networks  $T_R$  may vary significantly from channel to channel and layer to layer.

Referring to the subscripting procedure of figure C, the array of  $T_R$  values is expressed as

$$T_{Rn,l} = K_{n,l}/U_{n,l} \quad (72)$$

The Algorithm:

Given initial values of the channel and junction variables,

1. Compute  $\frac{\Delta U_{n,l}}{\Delta t}$  for each channel layer at time  $t=0$
2. Project ahead to predict new  $U_{n,l}$  at time  $t + \frac{\Delta t}{2}$ .
3. Compute  $Q_{n,l}$ ,  $\tau_{n,l}$  for time  $t + \frac{\Delta t}{2}$
4. Compute  $\frac{\Delta W_{I,l}}{\Delta t}$  for each junction layer.
5. Project ahead to predict new  $W_{I,l}$  at time  $t + \frac{\Delta t}{2}$
6. Compute  $q_{I,l}$  for time  $t + \frac{\Delta t}{2}$
7. Compute  $\frac{\Delta y_{I,l}}{\Delta t}$  for each junction layer based on flows from  
3. and 6.
8. Compute new  $H_{I,l}$ ,  $S_{n,l}$  and  $P_{aI,l}$  for time  $t + \frac{\Delta t}{2}$
9. Compute new  $\frac{\Delta U_{n,l}}{\Delta t}$  and average with the values of step 1.
10. Compute  $U_{n,l}$  for time  $t + \Delta t$ .
11. Compute  $Q_{n,l}$ ,  $\tau_{n,l}$  for time  $t + \Delta t$ .
12. Compute new  $\frac{\Delta W_{I,l}}{\Delta t}$  and average with those of step 4.
13. Compute  $q_{I,l}$  for time  $t + \Delta t$ .

14. Compute  $\frac{\Delta y_{I,l}}{\Delta t}$  for each junction based on flows from 11 and 13.
15. Compute  $H_{I,l}$ ,  $S_{n,l}$ , and  $P_{aI,l}$  for time  $t + \Delta t$ .
16. Compute new values for  $A_{n,l}$ ,  $d_{n,l}$ ,  $V_{I,l}$
17. Transfer arrays to be initial conditions for next time step.
18. Compute values of  $1/T_{Rn,l}$  for each channel layer and add to previous value to form  $t + \Delta t$ 

$$\sum_t 1/T_{Rn,l}$$
19. Examine  $\sum 1/T_{Rn,l}$  to see if it is greater than or equal to 1.0.
  - a.) if  $\sum 1/T_{Rn,l}$  is less than 1.0, compute and store  $Q_{n,l} \rho_{n,l}$  using old junction layer density and subtract from  $Q_{n,l} \rho_{n,l}$  computed using present junction layer density. Store the difference.
  - b.) if  $\sum 1/T_{Rn,l}$  is greater than or equal to 1.0, compute  $Q_{n,l} \rho_{n,l}$  and add all the differences computed in (a). Set the  $\sum 1/T_{Rn,l}$  term equal to 0.0.
20. At each junction layer compute the new junction density, and store for next cycle.
21. Return to step 1.

Steps 18 through 20 indicate one way of using the  $T_{R_n, \ell}$  terms. What is being done is to insure that the affect of a new upstream junction layer density will not be felt in the downstream junction layer until the flow carrying that new density has had time to arrive at the downstream junction layer.

An alternative method which may be suitable for small scale networks is to compute the network average of  $1/T_{R_n, \ell}$  and bypass the density computation for all junctions until the  $1/T_{R_n, \ell}$  term is greater than or equal to 1.0. The density computations can then be made using averaged flows over the total time period. This second method is the most efficient from a memory and time standpoint.

Conclusions:

The equations of motion and continuity have been derived and applied to stratified estuary networks. An algorithm has been presented which can be used to solve the equations. This algorithm must not be considered complete or totally correct in its present form. Nor is the treatment of the various terms (i.e., friction, mass transfer) necessarily in final form.

As is commonly the case, the methods and techniques used by the computer program will reflect the limitations in time and space imposed by the machine and the numerical difficulties encountered by the programmer. In addition, a sensitivity analysis on each parameter may indicate areas where simplifications can be made or where elaboration is needed.

It must be emphasized here that this method is to give an hydraulic solution. Although density is a primary factor in the solution, no indication is given as to the type of source of the densities. Once the vertical and horizontal flow patterns are known for the system, a quality model can predict the concentrations of various constituents at each junction layer.

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